

# Aharonov-Bohm Effect with $\delta$ -type Interaction

L. Dąbrowski<sup>1</sup> and P. Šťovíček<sup>2</sup>

<sup>1</sup>*S.I.S.S.A., 34014 Trieste, via Beirut 2-4, Italy*

<sup>2</sup>*Department of Mathematics, Faculty of Nuclear Science, CTU,  
Trojanova 13, 120 00 Prague, Czech Republic*

## Abstract

A quantum particle interacting with a thin solenoid and a magnetic flux is described by a five-parameter family of Hamilton operators, obtained via the method of self-adjoint extensions. One of the parameters, the value of the flux, corresponds to the Aharonov-Bohm effect; the other four parameters correspond to the strength of a singular potential barrier. The spectrum and eigenstates are computed and the scattering problem is solved.

## 0 Introduction

The purpose of this paper is to obtain and study the most general family of operators which describe the essential features of a quantum mechanical particle under the joint effect of the electromagnetic potential due to a flux  $\phi$  together with the potential barrier supported on the infinite thin shielded solenoid.

Our initial task is to provide a class of well defined operators corresponding to the formal differential (plus distributional) expression

$$-\sum_{\ell=1}^3(\partial/\partial x_\ell - A(\partial_\ell))^2 - \lambda\delta(r) ,$$

where  $x_1, x_2, x_3$  are the standard coordinates in  $\mathbb{R}^3$ ,  $A = i(\phi/2\pi r^2)(-x_2 dx_1 + x_1 dx_2)$  is a pure gauge potential and  $r = ((x_1)^2 + (x_2)^2)^{1/2}$ . For that aim, we first reduce the problem to two dimensions by making use of translational symmetry with respect to the coordinate  $x_3$ . Let  $r, \theta$  be, respectively, the radial and angular coordinate in  $\mathbb{R}^2$ ,  $0 \leq r$  and  $0 \leq \theta \leq 2\pi$ . Set  $\alpha = -\phi/2\pi$ . We concentrate on the case when  $\alpha \notin \mathbb{Z}$  and owing to the gauge symmetry  $A' = A + e^{-in\theta} de^{in\theta}$ ,  $n \in \mathbb{Z}$ , we can restrict ourselves to the case

$$\alpha \in ]0, 1[ . \quad (1)$$

The method we adopt in this paper is based on self adjoint extensions of symmetric operators. From this perspective, we try to combine two well known cases which were already extensively discussed in the literature. The first one, with  $\alpha = 0$ , corresponds to the so called point interaction (cf. [3]) and was studied in detail in [2]. An operator in a one-parameter family is defined on a domain which is characterized by a linear relation between certain coefficients which are built up from the asymptotic behaviour, as  $r \rightarrow 0$ , of (singular) wave functions. The starting point was a symmetric operator with a domain formed by wave functions with supports separated from the origin. The deficiency indices turned out to be (1,1).

The second case, with  $\lambda = 0$ , corresponds to the pure Aharonov-Bohm (A-B) potential [1] and was investigated many times on different levels (see [7], [9], [4], [8], [10]). The generalized eigenfunctions are required to belong to  $H_{\text{loc}}^{2,2}(\mathbb{R}^2 \setminus \{0\})$  and at the origin the regular condition

$$\lim_{r \rightarrow 0_+} f(r, \theta) = 0 \quad (2)$$

is imposed. As the Hamiltonian enjoys rotational symmetry the generalized eigenfunctions are known and the scattering problem is solved explicitly.

However, it has been known already for some time that when decomposing the Hilbert space into a direct sum,

$$L^2(\mathbb{R}^2, d^2x) \simeq L^2(\mathbb{R}_+, r dr) \otimes L^2([0, 2\pi], d\theta) = \oplus_{m=-\infty}^{\infty} L^2(\mathbb{R}_+, r dr) \otimes \chi_m , \quad (3)$$

where

$$\chi_m(\theta) = (2\pi)^{-1/2} e^{im\theta} , \quad (4)$$

then the A-B operator decomposes correspondingly and in the channels  $m = -1$  and  $m = 0$  the boundary condition (2) is not the most general one. But since the other boundary conditions admit wave functions which are singular at the origin they were usually ruled out.

To our opinion, it makes good sense to consider the most general case and hence to allow even a sort of interaction between the two channels. Thus we apply to the pure A-B Hamiltonian exactly the same procedure which was used in the case of point interactions. The deficiency indices one obtains this way are (2,2) and this indicates clearly that the result is not simply a superposition of the two special cases.

## 1 Five-parameter family of Hamilton operators and their resolvents

In order to get operators which can be consistently interpreted as describing the physical situation we are interested in, we start with the case of pure A-B effect and introduce the point interaction at  $0 \in \mathbb{R}^2$  in the usual way. Namely, first we consider the restriction of the self-adjoint pure A-B operator  $H$  to the space of functions with supports outside of  $\{0\}$ , obtaining thus a closable symmetric operator. Then we shall find all possible self-adjoint extensions of its closure  $\tilde{H}$ .

The adjoint  $\tilde{H}^*$  is defined as the differential operator  $-(\nabla - A(\nabla))^2$  on the domain

$$\psi \in \mathcal{D}(\tilde{H}^*) \iff \psi \in L^2(\mathbb{R}^2) \cap H_{\text{loc}}^{2,2}(\mathbb{R}^2 \setminus \{0\}) \text{ and } (\nabla - A(\nabla))^2 \psi \in L^2(\mathbb{R}^2). \quad (5)$$

On general grounds,  $\tilde{H}$  has equal deficiency indices. To find the corresponding deficiency spaces we employ the decomposition (3). Since the orthogonal projection onto  $L^2(\mathbb{R}_+) \otimes \chi_m$  commutes with  $\tilde{H}$  on  $\mathcal{D}(\tilde{H})$ , we can solve the eigenvalue problem

$$\tilde{H}^* f = k^2 f, \quad (6)$$

with  $k = e^{i\pi/4}$  and  $k = e^{i3\pi/4}$ , in each sector  $m$  of the angular momentum. Setting  $f(r, \theta) = g(r)\chi_m(\theta)$ , (6) becomes

$$-(\partial^2/\partial r^2 + 1/r \partial/\partial r + (m + \alpha)^2/r^2)g = k^2 g, \quad (7)$$

which, by the standard substitution  $r \rightarrow kr$ , leads to the Bessel equation.

Next, selecting in the two-dimensional space of solutions the one which vanishes at the infinity, we arrive at the Henkel functions

$$g(r) = H_{|m+\alpha|}^{(1)}(kr). \quad (8)$$

To ensure the integrability we still have to control the asymptotics as  $r \rightarrow 0_+$ . The case  $\alpha = 0$  is known; the  $L^2$  solutions exist only in the sector  $m = 0$  and thus the

the deficiency indices are  $(1, 1)$ . Assuming now that  $0 < \phi < 2\pi$  and recalling the asymptotics

$$H_\nu^{(1)}(z) = -\frac{2^\nu i}{\sin \pi \nu \Gamma(1-\nu)} z^{-\nu} + \frac{2^{-\nu} i e^{-i\pi \nu}}{\sin \pi \nu \Gamma(1+\nu)} z^\nu + O(z^{-\nu+2}), \quad (9)$$

as  $z \rightarrow 0$ , the integrability at 0 means that  $2|m + \alpha| - 1 < 1$ , which selects precisely two angular momentum sectors:  $m = -1$  and  $m = 0$ . Thus the deficiency indices of  $\tilde{H}$  are  $(2, 2)$  and the deficiency space  $\mathcal{N}_i$  is spanned by  $f_i^1$  and  $f_i^2$  given by

$$f_i^1(r, \theta) = N_1 H_{1-\alpha}^{(1)}(kr) e^{-i\theta}, \quad f_i^2(r, \theta) = N_2 H_\alpha^{(1)}(kr), \quad (10)$$

where  $k = e^{i\pi/4}$  ( $k = \sqrt{i}$  with  $\text{Im } k > 0$ ) and the normalization constants  $N_1, N_2$  will be determined later on.

This means that all self-adjoint extensions are in one-to-one correspondence with the elements of the unitary group  $U(2)$  and are determined by boundary conditions at the origin. We treat them in detail in the next section. Thus we get, apart of  $\alpha$  characterizing the magnetic flux, four additional parameters.

It is now straightforward to determine the domain  $\mathcal{D}(\tilde{H})$ . As  $\tilde{H} = \tilde{H}^{**}$  it holds true that

$$\psi \in \mathcal{D}(\tilde{H}) \iff \psi \in \mathcal{D}(\tilde{H}^*) \quad \text{and} \quad \langle \psi, \tilde{H}^* \varphi \rangle = \langle \tilde{H}^* \psi, \varphi \rangle, \quad \forall \varphi \in \mathcal{N}_i + \mathcal{N}_{-i}.$$

Consequently we find that  $g(r) \otimes \chi_m(\theta) \in \mathcal{D}(\tilde{H})$ , with  $m = -1, 0$ , if and only if  $g \in L^2(\mathbb{R}_+, r dr) \cap H_{\text{loc}}^{2,2}([0, +\infty[)$ ,  $(\partial_r^2 + r^{-1} \partial_r + r^{-2}(m + \alpha)^2)g \in L^2(\mathbb{R}_+, r dr)$ , and

$$\lim_{r \rightarrow 0+} r W(g, h_\pm) = 0, \quad (11)$$

where  $h_\pm = H_{|m+\alpha|}^{(1)}(\sqrt{\pm i} r)$  (with  $\text{Im } \sqrt{\pm i} > 0$ ) and the symbol  $W(g, h)$  stands for the Wronskian,

$$W(g, h) := \bar{g} \partial_r h - \bar{h} \partial_r g.$$

At this point, let us make a short digression and recall a useful formula comparing resolvents of two self-adjoint extensions. It is stated in the framework of Krein's approach to the theory of self-adjoint extensions and was presented originally in [5]. Let us consider a general situation when a closed symmetric operator  $X$  is given and  $A_0$  is a self-adjoint extension of  $X$ ,  $X \subset A_0 = A_0^* \subset X^*$ . Assume that the deficiency indices of  $X$  are  $(d, d)$ ,  $d < \infty$ . Set

$$\mathcal{N}_z := \text{Ker}(X^* - z), \quad R_z^0 := (A_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (12)$$

The following facts are well known and easy to check. First,

$$\psi \in \mathcal{N}_w \implies \psi + (z - w) R_z^0 \psi = (A_0 - w) R_z^0 \psi \in \mathcal{N}_z. \quad (13)$$

Second, if  $U_z^0 : \mathcal{N}_z \rightarrow \mathcal{N}_{\bar{z}}$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ , is the unitary mapping defining the self-adjoint extension  $A_0$  then

$$\forall \psi \in \mathcal{N}_z, \quad U_z^0 \psi = -(A_0 - z)(A_0 - \bar{z})^{-1} \psi. \quad (14)$$

Fix  $w \in \mathbb{C} \setminus \mathbb{R}$  and a basis  $\{f_w^\ell\}_\ell$  in  $\mathcal{N}_w$ . Set

$$f_z^\ell = f_w^\ell + (z - w)R_z^0 f_w^\ell = (A_0 - w)R_z^0 f_w^\ell, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (15)$$

Since  $(A_0 - w)R_z^0$  is injective and  $\dim \mathcal{N}_w = \dim \mathcal{N}_z = d$ ,  $\{f_z^1, \dots, f_z^d\}$  is a basis in  $\mathcal{N}_z$ . Suppressing the superscript  $\ell$  one can verify readily that

$$\forall z, z' \in \mathbb{C} \setminus \mathbb{R}, \quad f_z = f_{z'} + (z - z')R_z^0 f_{z'}, \quad (16)$$

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad U_z^0 f_z = -f_{\bar{z}}. \quad (17)$$

Thus, in order to reproduce the vector-valued function  $f_z$ , one can take any  $z' \in \mathbb{C} \setminus \mathbb{R}$  instead of  $w$ . Furthermore, the matrix of  $U_z^0$  in the bases  $\{f_z^\ell\}_\ell, \{f_{\bar{z}}^\ell\}_\ell$  equals  $-1$ .

To proceed further let us introduce a matrix  $P(z, w) = (P^{jk}(z, w))$  of scalar products relating two spaces  $\mathcal{N}_z$  and  $\mathcal{N}_w$ ,

$$P^{jk}(z, w) := \langle f_z^j, f_w^k \rangle. \quad (18)$$

One finds that

$$P(z, z) = P(\bar{z}, \bar{z}), \quad (19)$$

and if  $U$  is any matrix expressed in the bases  $\{f_z^\ell\}, \{f_{\bar{z}}^\ell\}$ , and representing a unitary mapping  $\mathcal{N}_z \rightarrow \mathcal{N}_{\bar{z}}$  then

$$P(z, z) = U^* P(\bar{z}, \bar{z}) U. \quad (20)$$

Furthermore, if  $\mathcal{P}_z$  is the orthogonal projector onto  $\mathcal{N}_z$  then

$$\mathcal{P}_z = \sum_{jk} \left( P(z, z)^{-1} \right)^{jk} f_z^j \langle f_z^k, \cdot \rangle. \quad (21)$$

Our primary interest is to compare  $A_0$  with another self-adjoint extension  $A$  of  $X$  corresponding to a family of unitary mappings  $U_z : \mathcal{N}_z \rightarrow \mathcal{N}_{\bar{z}}$ . Krein's formula tells us that

$$R_z - R_z^0 = (\bar{z} - z)^{-1} \mathcal{P}_z^* (U_{\bar{z}} - U_{\bar{z}}^0) \mathcal{P}_{\bar{z}}, \quad (22)$$

with the symbol  $\mathcal{P}_z^*$  standing for the embedding of  $\mathcal{N}_z$  into the global Hilbert space. This means that there exists a family of  $d \times d$  matrices,  $M_z = (M_z^{jk})$ , such that

$$R_z - R_z^0 = \sum_{jk} M_z^{jk} f_z^j \langle f_{\bar{z}}^k, \cdot \rangle. \quad (23)$$

We claim that

$$M_z - M_w = (z - w) M_z P(\bar{z}, w) M_w, \quad (24)$$

$$(\bar{w} - w) M_w = (U^{-1} + 1) P(w, w)^{-1}, \quad (25)$$

where  $U$  is the matrix of  $U_w : \mathcal{N}_w \rightarrow \mathcal{N}_{\bar{w}}$  in the bases  $\{f_w^j\}$  and  $\{f_{\bar{w}}^j\}$ . The proof is quite straightforward and relies on Krein's formula, the first resolvent identity and the explicit expression for  $\mathcal{P}_z$ . Provided  $M_w$  and  $M_z$  are invertible one can also write

$$M_w^{-1} - M_z^{-1} = (z - w) P(\bar{z}, w). \quad (26)$$

What we shall need in the sequel is the particular choice of  $w = -\imath$ . In this case,

$$M_z^{-1} = 2\imath P(\imath, \imath)(U + 1)^{-1} - (z + \imath)P(\bar{z}, -\imath), \quad (27)$$

where this time  $U$  is the matrix of  $U_\imath : \mathcal{N}_\imath \rightarrow \mathcal{N}_{-\imath}$  in the above specified bases.

Next we apply this general procedure to our problem, with  $A_0 \equiv H$  – the pure A-B operator. Thus, for  $z \in \mathbb{C} \setminus \mathbb{R}_+$ , we choose in  $\mathcal{N}_z$  a particular basis which depends holomorphically on  $z$  by

$$f_z^\ell = (1 + (z - \imath)R_z)f_\imath^\ell, \quad \ell = 1, 2, \quad (28)$$

where  $R_z$  is the resolvent of the pure A-B operator defined by its integral kernel (Green function)

$$G_z(r, \theta; r', \theta') = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{\imath m(\theta - \theta')} \int_0^\infty (k^2 - z)^{-1} J_{|m+\alpha|}(kr) J_{|m+\alpha|}(kr') k dk. \quad (29)$$

Recalling that

$$H_\nu^{(1)}(z) = \frac{2}{\pi \imath} e^{-\imath \pi \nu / 2} K_\nu(-\imath z), \quad (30)$$

where

$$K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh \nu t \, dt, \quad (31)$$

and using the formulae 8.13(3) and 8.5(12) of [11],

$$\int_0^\infty K_\mu(ay) J_\mu(xy) y \, dy = \left(\frac{x}{a}\right)^\mu (x^2 + a^2)^{-1}, \quad (32)$$

$$\int_0^\infty \frac{x^\mu}{x^2 + a^2} J_\mu(yx) x \, dx = a^\mu K_\mu(ay), \quad \mu \in ]-1, 3/2[, \operatorname{Re} a > 0, \quad (33)$$

we have

$$\begin{aligned} f_z^1(r, \theta) &= N_1 e^{-\imath \pi (1-\alpha)/4} (\sqrt{z})^{1-\alpha} H_{1-\alpha}^{(1)}(\sqrt{z}r) e^{-\imath \theta}, \\ f_z^2(r, \theta) &= N_2 e^{-\imath \pi \alpha/4} (\sqrt{z})^\alpha H_\alpha^{(1)}(\sqrt{z}r), \end{aligned} \quad (34)$$

where  $z \in \mathbb{C} \setminus \mathbb{R}_+$ ,  $\operatorname{Im} \sqrt{z} > 0$ . Particularly,

$$f_{-\imath}^1(r, \theta) = N_1 e^{\imath \pi (1-\alpha)/2} H_{1-\alpha}^{(1)}(kr) e^{-\imath \theta}, \quad f_{-\imath}^2(r, \theta) = N_2 e^{\imath \pi \alpha/2} H_\alpha^{(1)}(kr), \quad (35)$$

where  $k = \sqrt{-\imath} = e^{\imath 3\pi/4}$ .

A word of warning should be said here. We use the branch  $(e^{\imath \varphi})^\nu = e^{\imath \varphi \nu}$ , for  $0 \leq \varphi < 2\pi$ , which differs from the usual choice made in surveys like [6], [11].

Next we compute the matrix  $P(z, z')$ ,

$$P^{\ell, m}(z, z') := \langle f_z^\ell, f_{z'}^m \rangle = P^{\ell, \ell}(z, z') \delta_{\ell, m}, \quad \ell, m = 1, 2. \quad (36)$$

Since  $\overline{K_\nu(z)} = K_\nu(\bar{z})$  and using the formula 6.521 of [6],

$$\int_0^\infty K_\nu(ar) K_\nu(br) r \, dr = \frac{\pi(ab)^{-\nu}(a^{2\nu} - b^{2\nu})}{2 \sin(\pi \nu)(a^2 - b^2)}, \quad (37)$$

we find that

$$P^{\ell,\ell}(z, z') = N_\ell^2 \frac{4e^{-i\pi\nu}}{\sin(\pi\nu)} \frac{(\bar{z})^\nu - (z')^\nu}{z - z'}, \quad (38)$$

where

$$\nu = \begin{cases} 1 - \alpha & \text{if } \ell = 1 \\ \alpha & \text{if } \ell = 2 \end{cases}.$$

Making use of the identity

$$\int_0^\infty |H_\nu^{(1)}(kr)|^2 r dr = \left( \pi \cos(\pi\nu/2) \right)^{-1}, \quad k = \sqrt{\pm i}, \quad \text{Im } k > 0, \quad (39)$$

we choose now

$$N_1 = 2^{-1/2} \sin^{1/2}(\pi\alpha/2), \quad N_2 = 2^{-1/2} \cos^{1/2}(\pi\alpha/2). \quad (40)$$

In this case the basis  $\{f_i^\ell\}$  (10), as well as the basis  $\{f_{-i}^\ell\}$ , is orthonormal, i.e., in the matrix notation,

$$P(i, i) = P(-i, -i) = I, \quad (41)$$

where  $I$  is the  $2 \times 2$  unit matrix. Moreover, introducing the matrix

$$D = \text{diag}(1 - \alpha, \alpha), \quad (42)$$

we also have

$$P(\bar{z}, -i) = (z + i)^{-1} \sin^{-1}(\pi D/2) e^{-i\pi D} (e^{3i\pi D/2} - z^D). \quad (43)$$

We conclude that all self-adjoint extensions  $H^U$  of  $\tilde{H}$  are bijectively labelled by  $2 \times 2$  unitary matrices  $U$  by means of

$$\mathcal{D}(H^U) := \{\psi; \psi = \tilde{\psi} + \sum_\ell c_\ell \psi^\ell\}, \quad (44)$$

where  $\tilde{\psi} \in D(\tilde{H})$  and

$$\psi^\ell = f_i^\ell + \sum_m f_{-i}^m U^{m,\ell}, \quad \ell = 1, 2,$$

and

$$H^U \psi = \tilde{H} \tilde{\psi} + \sum_\ell c_\ell (i f_i^\ell - i \sum_m f_{-i}^m U^{m,\ell}). \quad (45)$$

According to the above discussion,  $U = -1$  corresponds to the pure A-B operator  $H$ . Moreover, diagonal  $U$  describe the extensions preserving the angular momentum (which otherwise is non-conserved).

## 2 Boundary conditions

The family of operators  $H^U$ , defined so far abstractly, can be equivalently characterized as differential operators with some well specified boundary conditions. For this purpose, we introduce four linear functionals  $\Phi_a^n$ ,  $n = 1, 2$ ,  $a = 1, 2$ , corresponding to the two critical angular sectors and to the first two leading terms giving the asymptotic behaviour of the radial part of  $\psi$  as  $r \rightarrow 0$ . We define

$$\begin{aligned}\Phi_1^1(\psi) &:= \lim_{r \rightarrow 0} r^{1-\alpha} \int_0^{2\pi} \psi(r, \theta) e^{i\theta} d\theta / 2\pi, \\ \Phi_2^1(\psi) &:= \lim_{r \rightarrow 0} r^{-1+\alpha} \left[ \int_0^{2\pi} \psi(r, \theta) e^{i\theta} d\theta / 2\pi - r^{-1+\alpha} \Phi_1^1(\psi) \right], \\ \Phi_1^2(\psi) &:= \lim_{r \rightarrow 0} r^\alpha \int_0^{2\pi} \psi(r, \theta) d\theta / 2\pi, \\ \Phi_2^2(\psi) &:= \lim_{r \rightarrow 0} r^{-\alpha} \left[ \int_0^{2\pi} \psi(r, \theta) d\theta / 2\pi - r^{-\alpha} \Phi_1^2(\psi) \right].\end{aligned}\tag{46}$$

This definition is, of course, dictated by the asymptotic behaviour of Hankels functions (cf. (9)). So for  $\psi \in \mathcal{D}(H^U)$ , the part of  $\psi$  which is singular or becomes singular after differentiation by  $\partial_r$  is given by

$$\left( \Phi_1^1(\psi) r^{-1+\alpha} + \Phi_2^1(\psi) r^{1-\alpha} \right) e^{-i\theta} + \Phi_1^2(\psi) r^{-\alpha} + \Phi_2^2(\psi) r^\alpha.$$

Let us first check the symmetry condition  $\langle \varphi_1, H^U \varphi_2 \rangle = \langle H^U \varphi_1, \varphi_2 \rangle$ , for  $\varphi_1, \varphi_2 \in \mathcal{D}(H^U)$ . The integration by parts gives ( $W$  = Wronskian)

$$\lim_{r \rightarrow 0+} \int_0^{2\pi} r W(\varphi_1, \varphi_2) d\theta = 0.$$

Only the singular parts of  $\varphi_1, \varphi_2$  can contribute and thus one arrives at

$$\Phi_1(\varphi_1)^* D \Phi_2(\varphi_2) = \Phi_1(\varphi_2)^* D \Phi_2(\varphi_1), \quad \forall \varphi_1, \varphi_2 \in \mathcal{D}(H^U),\tag{47}$$

where we have introduced

$$\Phi_a(\psi) := \begin{pmatrix} \Phi_a^1(\psi) \\ \Phi_a^2(\psi) \end{pmatrix}, \quad a = 1, 2,\tag{48}$$

and  $D$  was defined in (42).

Next we apply the functionals  $\Phi_a^n$  to the functions  $f_{\pm i}^\ell$ . Namely, introduce four matrices  $\Phi_{ab}$ , where the label  $a = 1, 2$  refers to the first (respectively the second) leading coefficient, and  $b = \pm$  refers to  $\pm$  in  $\mathcal{N}_{\pm i}$ . They are defined by

$$(\Phi_{ab})^{n\ell} := \Phi_a^n(f_{bi}^\ell); \tag{49}$$

so the rows of these matrices are numbered by the angular momentum  $n$  (for the sake of convenience we shifted the index by 2,  $n = m + 2 \in \{1, 2\}$ ) and the columns



by  $\ell$ , corresponding to the basis in  $\mathcal{N}_{\pm\iota}$ . In view of the asymptotic expansion of the functions  $f_{b\ell}^\ell$  (cf. (10) and (35)), they read explicitly

$$\begin{aligned}\Phi_{1,+} &= \Phi_{1,-} \\ &= \frac{-\iota 2^{-1/2}}{\sin(\pi\alpha)} \cos^{1/2}(\pi D/2) 2^D \Gamma(1-D)^{-1} \exp(-\iota\pi D/4), \\ \Phi_{2,+} &= \frac{\iota 2^{-1/2}}{\sin(\pi\alpha)} \cos^{1/2}(\pi D/2) 2^{-D} \Gamma(1+D)^{-1} \exp(-\iota 3\pi D/4), \\ \Phi_{2,-} &= \frac{\iota 2^{-1/2}}{\sin(\pi\alpha)} \cos^{1/2}(\pi D/2) 2^{-D} \Gamma(1+D)^{-1} \exp(\iota\pi D/4).\end{aligned}\tag{50}$$

Here and everywhere in what follows we use the obvious notation: if a function  $F$  is well defined on the set  $\{1-\alpha, \alpha\}$  then

$$F(D) := \text{diag}(F(1-\alpha), F(\alpha)).$$

From the definition (44) of  $\mathcal{D}(H^U)$  follows that  $\psi \in L^2(\mathbb{R}^2)$  belongs to the domain of  $H^U$  if and only if

$$\begin{pmatrix} \Phi_1(\psi) \\ \Phi_2(\psi) \end{pmatrix} \in \text{Ran} \begin{pmatrix} \Phi_{1,+} + \Phi_{1,-}U \\ \Phi_{2,+} + \Phi_{2,-}U \end{pmatrix}.\tag{51}$$

Inversely, assume that we are given a couple of  $2 \times 2$  matrices  $X_1, X_2$  such that  $\text{rank}(X_1^t, X_2^t) = 2$ , and consider the boundary condition

$$\begin{pmatrix} \Phi_1(\psi) \\ \Phi_2(\psi) \end{pmatrix} \in \text{Ran} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.\tag{52}$$

The symmetry property (47) leads to the requirement

$$X_1^* D X_2 = X_2^* D X_1.\tag{53}$$

In fact, relying on the explicit form of the matrices  $\Phi_{ab}$ , one can show quite straightforwardly that for any couple  $X_1, X_2$  with the above properties there exist exactly one  $2 \times 2$  unitary matrix  $U$  and  $Y \in GL(2, \mathbb{C})$  such that

$$X_1 Y = \Phi_{1,+} + \Phi_{1,-}U, \quad X_2 Y = \Phi_{2,+} + \Phi_{2,-}U.\tag{54}$$

On the contrary, if  $U$  is unitary then  $X_a = \Phi_{a,+} + \Phi_{a,-}U$ ,  $a = 1, 2$ , verify (53) and  $\text{rank}(X_1^t, X_2^t) = 2$ . This way we have rederived a well known result that all self-adjoint extensions of  $\tilde{H}$  are in one-to-one correspondence with points of a real 4-dimensional submanifold of the Grassmann manifold  $G_2(\mathbb{C}^4)$  determined by the equation (53).

One can rewrite the boundary condition (52) in a more convenient form when making use of the biholomorphic diffeomorphism  $G_2(\mathbb{C}^4) \rightarrow G_2(\mathbb{C}^4)^\sharp$ . Here  $G_2(\mathbb{C}^4)^\sharp$  stands for the Grassmann manifold in the space dual to  $\mathbb{C}^4$ . The points of  $G_2(\mathbb{C}^4)^\sharp$

are represented by couples of matrices  $A_1, A_2 \in \text{Mat}(2, \mathbb{C})$ , with  $\text{rank}(A_1, A_2) = 2$ , modulo the left action of  $GL(2, \mathbb{C})$ . The diffeomorphism is given by

$$\begin{aligned} \text{Ran} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in G_2(\mathbb{C}^4) &\mapsto \text{Ran} \begin{pmatrix} A_1^t \\ A_2^t \end{pmatrix} \in G_2(\mathbb{C}^4), \\ \text{where } A_1 X_1 + A_2 X_2 &= 0. \end{aligned} \quad (55)$$

The real submanifold of  $G_2(\mathbb{C}^4)$  determined by (53) is mapped bijectively onto the real 4-dimensional submanifold of  $G_2(\mathbb{C}^4)^\sharp$  determined by

$$A_1 D^{-1} A_2^* = A_2 D^{-1} A_1^*. \quad (56)$$

We conclude that each self-adjoint extension of  $\tilde{H}$  is determined by a boundary condition of the type

$$A_1 \Phi_1(\psi) + A_2 \Phi_2(\psi) = 0, \quad (57)$$

where  $A_1, A_2 \in \text{Mat}(2, \mathbb{C})$  verify (56) and  $\text{rank}(A_1, A_2) = 2$ . Two couples  $(A_1, A_2)$  and  $(A'_1, A'_2)$  define the same self-adjoint operator if and only if there exists  $Y \in GL(2, \mathbb{C})$  such that  $A'_1 = Y A_1$ ,  $A'_2 = Y A_2$ .

Let us now restrict ourselves to an open dense subset of the manifold of boundary conditions (57) which we obtain by fixing  $A_1 = I$  and setting  $A_2 = -\Lambda$ . The condition (57) means, of course, that

$$\begin{pmatrix} \Phi_1^1(\psi) \\ \Phi_1^2(\psi) \end{pmatrix} = \Lambda \begin{pmatrix} \Phi_2^1(\psi) \\ \Phi_2^2(\psi) \end{pmatrix}, \quad \forall \psi \in \mathcal{D}(H^U). \quad (58)$$

The restriction (56) then reads

$$D\Lambda = \Lambda^* D. \quad (59)$$

All matrices  $\Lambda$  obeying (59) can be parameterized by the aid of four real (or two real and one complex) parameters, namely

$$\Lambda = \begin{pmatrix} u & \alpha \bar{w} \\ (1-\alpha)w & v \end{pmatrix}, \quad \text{with } u, v \in \mathbb{R} \text{ and } w \in \mathbb{C}. \quad (60)$$

From (55) follows that one can choose  $X_1 = \Lambda$ ,  $X_2 = I$ . In virtue of (54), this leads to a relation between  $\Lambda$  and  $U$ ,

$$\Lambda = (\Phi_{1,+} + \Phi_{1,-}U)(\Phi_{2,+} + \Phi_{2,-}U)^{-1}, \quad (61)$$

provided the relevant matrix is invertible. We use the following parameterization of  $U$ ,

$$U = e^{i\omega} \begin{pmatrix} qe^{ia} & -(1-q^2)^{1/2}e^{-ib} \\ (1-q^2)^{1/2}e^{ib} & qe^{-ia} \end{pmatrix}, \quad (62)$$

where  $a, b, q, \omega \in \mathbb{R}$ ,  $0 \leq q \leq 1$ . The matrix  $(\Phi_{2,+} + \Phi_{2,-}U)$  is invertible exactly when  $d \neq 0$  where

$$d := \sin \omega + q \sin(a - \pi \alpha). \quad (63)$$

In this case, the parameters of the matrix  $\Lambda$  can be expressed explicitly in terms of  $a, b, q, \omega$ ,

$$\begin{aligned} u &= d^{-1} 2^{2-2\alpha} \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)} \left( \cos\left(\omega + \frac{\pi}{2}\alpha\right) + q \cos\left(a - \frac{\pi}{2}\alpha\right) \right), \\ \bar{w} &= d^{-1} \left( 2(1-q^2) \sin(\pi\alpha) \right)^{1/2} e^{-i(b - \frac{\pi}{2}\alpha - \frac{\pi}{4})}, \\ w &= d^{-1} \left( 2(1-q^2) \sin(\pi\alpha) \right)^{1/2} e^{i(b - \frac{\pi}{2}\alpha - \frac{\pi}{4})}, \\ v &= -d^{-1} 2^{2\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} \left( \sin\left(\omega - \frac{\pi}{2}\alpha\right) + q \sin\left(a - \frac{\pi}{2}\alpha\right) \right). \end{aligned} \quad (64)$$

Obviously,  $u = w = v = 0$  corresponds to the pure A-B case. Moreover, diagonal  $\Lambda$  describe the operators preserving the angular momentum ( $w$  is responsible for its non-conservation).

### 3 Spectrum and eigenspaces

In order to find the spectrum one can use Krein's formula for the resolvent  $R_z^U = (H^U - z)^{-1}$ ,

$$R_z^U = R_z + \sum_{k\ell} f_z^k M_z^{k\ell} \langle f_z^\ell, \cdot \rangle. \quad (65)$$

Using (41), (43) and (27) we get

$$M_z^{-1} = \sin^{-1}(\pi D/2) \left( z^D e^{-i\pi D} (U+1) - e^{i\pi D/2} U - e^{-i\pi D/2} \right) (U+1)^{-1}. \quad (66)$$

Since  $R_z^U$  is a rank two perturbation of  $R_z$ ,  $H^U$  and  $H$  have the same absolutely continuous spectrum, namely  $[0, \infty[$ .

The discrete spectrum is determined by the equation  $\det M_z^{-1} = 0$ , i.e.,

$$\det \left( p^{2D} (U+1) - e^{i\pi D/2} U - e^{-i\pi D/2} \right) = 0, \quad (67)$$

where we have introduced  $p > 0$  by  $p^2 = -z$ . According to the discussion below there are no non-negative eigenvalues. If  $d \neq 0$  (cf. (63)) then  $(e^{i\pi D/2} U - e^{-i\pi D/2})$  is invertible and owing to (61),

$$\begin{aligned} (U+1) \left( e^{i\pi D/2} U - e^{-i\pi D/2} \right)^{-1} &= - \cos^{-1/2}(\pi D/2) 2^{-D} \Gamma(1-D) e^{i\pi D/4} \Lambda \\ &\times \cos^{1/2}(\pi D/2) 2^{-D} \Gamma(1+D) e^{-i\pi D/4}. \end{aligned} \quad (68)$$

In this case, (67) is equivalent to

$$\det \left( \frac{\Gamma(1-D)}{\Gamma(1+D)} \left( \frac{p}{2} \right)^{2D} \Lambda + 1 \right) = 0. \quad (69)$$

Rewriting (69) in terms of the parameters  $u, v, w$  (cf. (60)) we get

$$\left( \left( \frac{p}{2} \right)^{2\alpha-2} + \frac{\Gamma(\alpha)}{\Gamma(2-\alpha)} u \right) \left( \left( \frac{p}{2} \right)^{-2\alpha} + \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} v \right) = |w|^2. \quad (70)$$

Consider the LHS of (70) as a function  $F(p)$  defined on  $]0, +\infty[$ . Since the RHS of (70) is always non-negative some elementary analysis gives immediately the number of solutions. The number of roots of  $F(p)$  equals 0 or 1 or 2. There is no root iff  $u \geq 0, v \geq 0$ , and there are two roots iff  $u < 0, v < 0$  (it may happen that the two roots coincide giving a multiple root). Denote by  $p_1$  the smallest root of  $F(p)$ , if any. In the case of two roots, let  $p_2$  be the greater one. Then clearly  $F(p)$  is decreasing on  $]0, p_1]$  from  $+\infty$  to zero and is increasing on  $[p_2, +\infty[$  from zero to the asymptotic value  $uv/\alpha(1-\alpha)$ .

We conclude that there are

$$\begin{aligned} \text{two eigenvalues} & \quad \text{if } u < 0, v < 0, \text{ and } \det \Lambda = uv - \alpha(1-\alpha)|w|^2 > 0, \\ \text{no eigenvalue} & \quad \text{if } u \geq 0, v \geq 0, \text{ and } \det \Lambda = uv - \alpha(1-\alpha)|w|^2 \geq 0, \\ \text{one eigenvalue} & \quad \text{otherwise.} \end{aligned} \quad (71)$$

We stress once more that all eigenvalues, if any, are negative. Again, it may happen (when  $w = 0$ ) that two eigenvalues coincide producing consequently a multiple eigenvalue.

For generic  $\alpha$  not much can be said about what are the solutions, except the case  $w = 0$  when  $p = 2 \left( -\frac{u\Gamma(\alpha)}{\Gamma(2-\alpha)} \right)^{2-2\alpha}$  if  $u < 0$  and/or  $p = 2 \left( -\frac{v\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \right)^{2\alpha}$  if  $v < 0$ , and the case  $u = v = 0$  when  $p = 2/|w|$ . An interesting particular case is  $\alpha = 1/2$  when we can give a complete answer about the values of the two solutions (the first case in (71)),

$$p_{\pm} = \frac{u+v \pm (|w|^2 + (u-v)^2)^{\frac{1}{2}}}{2(|w|^2/4 - uv)}, \quad (72)$$

and of the solution (the third case in (71)),

$$p = \frac{u+v + (|w|^2 + (u-v)^2)^{\frac{1}{2}}}{2(|w|^2/4 - uv)}. \quad (73)$$

Similar, but more complicated analysis can be also performed for  $\alpha = 1/3, 1/4$  and partially for other fractional values of  $\alpha$ .

As far as the eigenvectors are concerned, they have to be obtained, of course, as solutions of the differential equation (6) including the corresponding boundary conditions (58). First of all, it is clear that in the sectors of the angular momentum  $m \neq -1, 0$ , there is a complete system of generalized (and normalized) eigenfunctions coinciding with those of the pure A-B effect,

$$(2\pi)^{-1/2} J_{|m+\alpha|}(kr) e^{im\theta}, \quad m = \dots -3, -2, 1, 2, \dots, \quad k > 0. \quad (74)$$

Next we pass to the sectors  $m = -1$  and  $m = 0$ .

As far as the (true) eigenfunctions are concerned, the  $L^2$ -integrability condition at infinity restricts the eigenvalue  $k^2$  to  $k^2 < 0$ , and picks up a unique solution, up to a multiplicative constant, in each sector  $m = -1$  and  $m = 0$  (this means that both exponential growth and oscillatory behaviour at infinity are excluded). Hence setting as before  $k = \imath p$ , with  $p > 0$ , the eigenfunction must have the form

$$\xi H_{1-\alpha}^{(1)}(\imath pr) e^{-\imath\theta} + \eta H_{\alpha}^{(1)}(\imath pr) , \quad (75)$$

where  $\xi, \eta \in \mathcal{C}$ . The boundary conditions (58) lead to the following relation between  $\xi$  and  $\eta$ ,

$$\left( I + \left( \frac{\imath p}{2} \right)^D \Gamma(1-D) \wedge \Gamma(1+D)^{-1} e^{-\imath\pi D} \left( \frac{\imath p}{2} \right)^D \right) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 . \quad (76)$$

Setting the determinant of this system of linear equations to zero we recover the equation on eigenvalues (70). Any non-trivial solution  $(\xi, \eta)$  of (76) determines an eigenfunction in accordance with (75).

As far as the generalized eigenfunctions are concerned, the eigenvalue equation admits a four-parameter solution

$$(\xi J_{-1+\alpha}(kr) + \eta J_{1-\alpha}(kr)) e^{-\imath\theta} + \xi' J_{-\alpha}(kr) + \eta' J_{\alpha}(kr) , \quad (77)$$

where  $k > 0$  and  $\xi, \eta, \xi', \eta' \in \mathcal{C}$ .

In view of the asymptotics of  $J_{\nu}(z) \simeq \Gamma(1+\nu)^{-1} (z/2)^{\nu} (1 + O(z^2))$  and again by taking into account the boundary conditions (58) we find the relation

$$\begin{pmatrix} \xi \\ \xi' \end{pmatrix} = (k/2)^D \Gamma(1-D) \wedge \Gamma(1+D)^{-1} (k/2)^D \begin{pmatrix} \eta \\ \eta' \end{pmatrix} . \quad (78)$$

A possible choice is

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and in this way we obtain two independent solutions

$$\begin{aligned} b_1(k) &= (\xi_1 J_{-1+\alpha}(kr) + J_{1-\alpha}(kr)) e^{-\imath\theta} + \eta_1 J_{-\alpha}(kr) , \\ b_2(k) &= \xi_2 J_{-1+\alpha}(kr) e^{-\imath\theta} + \eta_2 J_{-\alpha}(kr) + J_{\alpha}(kr) , \end{aligned} \quad (79)$$

where

$$\begin{aligned} \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} &= (k/2)^D \Gamma(1-D) \wedge \Gamma(1+D)^{-1} (k/2)^D \\ &= \begin{pmatrix} u \frac{\Gamma(\alpha)}{\Gamma(2-\alpha)} \left( \frac{k}{2} \right)^{2-2\alpha} & \bar{w} \frac{k}{2} \\ w \frac{k}{2} & v \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \left( \frac{k}{2} \right)^{2\alpha} \end{pmatrix} . \end{aligned} \quad (80)$$

Set

$$N(k) = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} + \exp(\imath\pi D) . \quad (81)$$

Now we seek a pair of eigenfunctions which are complete and orthonormal in the generalized sense. In order to compute the scalar products of  $b_1(k)$  and  $b_2(k)$ , we need to know the integrals involving the products  $J_\mu(ay) J_\mu(xy)$  and  $J_{-\mu}(ay) J_\mu(xy)$ . Recalling the relation between the functions  $H_\mu^{(1)}$  and  $K_\mu$  (30) and using the limit value of (32) we have

$$\int_0^\infty H_\mu^{(1)}(ay) J_\mu(xy) y dy = \frac{1}{i\pi a} \left(\frac{x}{a}\right)^\mu \left(\frac{1}{x-a-i0} - \frac{1}{x+a}\right). \quad (82)$$

Next, with the help of the distributional identity  $(x - i0)^{-1} = \mathcal{P}(1/x) + i\pi\delta(x)$ , where  $\mathcal{P}$  denotes the principal part, we get two identities by taking the real and imaginary parts of (82),

$$\int_0^\infty J_\mu(ay) J_\mu(xy) y dy = \frac{1}{a} \delta(x-a), \quad (83)$$

$$\begin{aligned} \int_0^\infty J_{-\mu}(ay) J_\mu(xy) y dy &= \frac{\cos(\pi\mu)}{a} \delta(x-a) \\ &+ \frac{\sin(\pi\mu)}{\pi a} \left(\frac{x}{a}\right)^\mu \left(\mathcal{P}\left(\frac{1}{x-a}\right) - \frac{1}{x+a}\right). \end{aligned} \quad (84)$$

Applying (83)-(84) to the solutions (79) arranged in a row  $B(k) = (b_1(k), b_2(k))$ , we obtain the following  $2 \times 2$  matrix of scalar products,

$$\langle B(k'), B(k) \rangle = N(k')^* N(k) \frac{1}{k} \delta(k - k'). \quad (85)$$

We observe that

$$\det N(k) = \left(\frac{uv}{\alpha(1-\alpha)} - w\bar{w}\right) \frac{k^2}{4} + \frac{u \Gamma(\alpha) e^{i\pi\alpha} k^{2-2\alpha}}{2^{2-2\alpha} \Gamma(2-\alpha)} - \frac{v \Gamma(1-\alpha) e^{-i\pi\alpha} k^{2\alpha}}{2^{2\alpha} \Gamma(1+\alpha)} - 1 \quad (86)$$

equals minus the LHS of (69), with  $p$  being replaced by  $-ik$ , and thus, in view of our analysis of eq. (70),  $\det N(k)$  is nonvanishing for all  $k \geq 0$ . Therefore  $(g_1(k), g_2(k)) := B(k)N(k)^{-1}$ , being given by

$$\begin{aligned} g_1(k; r, \theta) &= \det^{-1} N(k) ((\xi_1 \eta_2 - \xi_2 \eta_1 + \xi_1 e^{i\pi\alpha}) J_{\alpha-1}(kr) e^{-i\theta} \\ &\quad + (\eta_2 + e^{i\pi\alpha}) J_{1-\alpha}(kr) e^{-i\theta} - \eta_1 J_\alpha(kr) + \eta_1 e^{i\pi\alpha} J_{-\alpha}(kr)), \\ g_2(k; r, \theta) &= \det^{-1} N(k) (-\xi_2 e^{-i\pi\alpha} J_{\alpha-1}(kr) e^{-i\theta} - \xi_2 J_{1-\alpha}(kr) e^{-i\theta} \\ &\quad + (\xi_1 - e^{-i\pi\alpha}) J_\alpha(kr) + (\xi_1 \eta_2 - \xi_2 \eta_1 - \eta_2 e^{-i\pi\alpha}) J_{-\alpha}(kr)), \end{aligned} \quad (87)$$

where  $\xi_1$ ,  $\xi_2$ ,  $\eta_1$  and  $\eta_2$  are defined by (80), form a complete orthonormal basis of generalized eigenvectors in the subspace corresponding to the absolutely continuous spectrum in the two considered sectors,

$$\langle g_j(k'), g_\ell(k) \rangle = \frac{1}{k} \delta(k - k') \delta_{j,\ell}. \quad (88)$$

## 4 Scattering

The existence of a complete and orthonormal basis of generalized eigenvectors is sufficient to show that the wave (Møller) operators  $W_{\pm} = \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH^0}$  exist and are complete. In fact, they can be exhibited explicitly as well as the scattering operator  $S = (W_+)^* W_-$ .

From what we have said so far it is evident that  $W_{\pm}$  and  $S$  preserve the sectors  $m \neq -1, 0$ , and there, they are exactly the same as in the pure A-B case; in particular  $S_m = e^{2i\delta_m}$ , where  $\delta_m = (|m| - |m + \alpha|)\pi/2$ . Thus we restrict ourselves to the subspace  $L^2(\mathbb{R}_+, r dr) \otimes (\mathbb{C}\chi_{-1} \oplus \mathbb{C}\chi_{-0})$ , of remaining sectors  $m = -1, 0$ , which is, of course, also preserved by all the relevant operators (if there is no danger of confusion we denote the restriction of an operator by the same symbol).

Using the basis  $g_1(k), g_2(k)$  we define a unitary operator  $\mathcal{F}$  from  $L^2(\mathbb{R}_+, k dk) \otimes \mathbb{C}^2$  to  $L^2(\mathbb{R}_+, r dr) \otimes (\mathbb{C}\chi_{-1} \oplus \mathbb{C}\chi_{-0})$ , by

$$\mathcal{F}[\tilde{\psi}] = \psi := \sum_{j=1}^2 \int_0^\infty g_j(k) \tilde{\psi}_j(k) k dk, \quad (89)$$

with the inverse  $\mathcal{F}^{-1}[\psi] = \tilde{\psi}$  given by

$$\tilde{\psi}_j(k) := \langle g_j(k), \psi \rangle. \quad (90)$$

The operator  $\mathcal{F}$  satisfies

$$e^{-itH} \mathcal{F}[\tilde{\psi}] = \mathcal{F}[e^{-itk^2} \tilde{\psi}] \quad (91)$$

We append the superscript ‘<sup>0</sup>’ to the relevant objects like  $g_j^0(k)$  or  $\mathcal{F}^0$  corresponding to the free case ( $\alpha = 0, \Lambda = 0$ ); particularly

$$g_1^0(k; r, \theta) = J_1(kr) e^{-i\theta}, \quad g_2^0(k; r, \theta) = J_0(kr). \quad (92)$$

Now we seek a pair of  $2 \times 2$  matrices  $\Omega_{\pm} = \Omega_{\pm}(k)$ , generally depending on  $k$  and acting in an obvious way as a multiplication operator on  $L^2(\mathbb{R}_+, k dk) \otimes \mathbb{C}^2$ , so that

$$W_{\pm} \mathcal{F}^0 = \mathcal{F} \Omega_{\pm}. \quad (93)$$

Then it follows that  $S \mathcal{F}^0 = \mathcal{F}^0 \Omega_+^* \Omega_-$ , and this means that  $\Sigma := \Omega_+^* \Omega_-$  is nothing but the scattering matrix in the momentum representation (restricted to the sectors  $m = -1, 0$ ).

We have to verify that

$$\lim_{t \rightarrow \pm\infty} \|e^{-itH^0} \mathcal{F}^0[\psi] - e^{-itH} \mathcal{F}[\Omega_{\pm}\psi]\| = 0. \quad (94)$$

Due to (91), the condition (94) means that

$$\lim_{t \rightarrow \pm\infty} \sum_j \int_0^\infty e^{-itk^2} |(g_j^0(k) - \sum_{\ell} g_{\ell}(k) \Omega_{\pm}(k)_{\ell j}) \psi_j(k)|^2 k dk = 0. \quad (95)$$

It is sufficient to prove (95) only for functions  $\psi_j(k)$  from the dense subspace  $C_0^\infty(\mathbb{R}_+) \subset L^2(\mathbb{R}_+, k dk)$ . By the stationary phase method, the convergence of such an integral, as  $t \rightarrow \pm\infty$ , will be established provided the coefficient standing in front of the term  $e^{\pm ikr}$  vanishes. In view of the known large  $x$  expansion

$$J_\mu(x) = (2\pi x)^{1/2} (e^{i(x-\pi\mu/2-\pi/4)} + e^{-i(x-\pi\mu/2-\pi/4)}) + O(x^{3/4}), \quad (96)$$

we obtain, by looking separately at the coefficients in front of  $e^{-i\theta}$  and 1, that

$$\begin{aligned} \Omega_+(k)^{-1} &= \text{diag}(-e^{-i\pi\alpha/2}, e^{i\pi\alpha/2}) \tilde{N}(k) N(k)^{-1} \\ &= \det^{-1} N(k) \text{diag}(-e^{-i\pi\alpha/2}, e^{i\pi\alpha/2}) \\ &\times \begin{pmatrix} \xi_1\eta_2 - \xi_2\eta_1 - e^{i\pi\alpha}(\eta_2 - \xi_1) - e^{2i\pi\alpha} & (e^{i\pi\alpha} - e^{-i\pi\alpha})\xi_2 \\ (e^{i\pi\alpha} - e^{-i\pi\alpha})\eta_1 & \xi_1\eta_2 - \xi_2\eta_1 - e^{-i\pi\alpha}(\eta_2 - \xi_1) - e^{-2i\pi\alpha} \end{pmatrix}, \end{aligned} \quad (97)$$

where

$$\tilde{N}(k) = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} + \exp(-i\pi D), \quad (98)$$

and also

$$\Omega_-(k) = \text{diag}(-e^{-i\pi\alpha/2}, e^{i\pi\alpha/2}). \quad (99)$$

Note that

$$\tilde{N}(k)^* \tilde{N}(k) = N(k)^* N(k) \quad (100)$$

and so  $\Omega_+(k)$  is unitary. Consequently, we can express (the entries of)  $\Sigma$  in terms of the parameters  $u, v, w$  (see (80)) as

$$\begin{aligned} \Sigma_{11} &= \det^{-1} N(k) \times \\ &\quad \left( e^{-i\pi\alpha} \left( \frac{uv}{\alpha(1-\alpha)} - |w|^2 \right) \frac{k^2}{4} + \frac{u\Gamma(\alpha)k^{2-2\alpha}}{2^{2-2\alpha}\Gamma(2-\alpha)} - \frac{v\Gamma(1-\alpha)k^{2\alpha}}{2^{2\alpha}\Gamma(1+\alpha)} - e^{i\pi\alpha} \right) \\ \Sigma_{12} &= -\det^{-1} N(k) i \sin(\pi\alpha) \bar{w}k \\ \Sigma_{21} &= -\det^{-1} N(k) i \sin(\pi\alpha) wk \\ \Sigma_{22} &= \det^{-1} N(k) \times \\ &\quad \left( e^{i\pi\alpha} \left( \frac{uv}{\alpha(1-\alpha)} - |w|^2 \right) \frac{k^2}{4} + \frac{u\Gamma(\alpha)k^{2-2\alpha}}{2^{2-2\alpha}\Gamma(2-\alpha)} - \frac{v\Gamma(1-\alpha)k^{2\alpha}}{2^{2\alpha}\Gamma(1+\alpha)} - e^{-i\pi\alpha} \right), \end{aligned} \quad (101)$$

where  $\det^{-1} N(k)$  is given by (86). Obviously,  $\Sigma$  is unitary.

Let us specialize these formulae to three particular cases.

(i) If  $w = 0$  (conserved angular momentum) then

$$\Sigma = \text{diag} \left( \frac{u\Gamma(\alpha)(k/2)^{2-2\alpha} - e^{i\pi\alpha}\Gamma(2-\alpha)}{e^{i\pi\alpha}u\Gamma(\alpha)(k/2)^{2-2\alpha} - \Gamma(2-\alpha)}, \frac{v\Gamma(1-\alpha)(k/2)^{2\alpha} + e^{-i\pi\alpha}\Gamma(1+\alpha)}{e^{-i\pi\alpha}v\Gamma(1-\alpha)(k/2)^{2\alpha} + \Gamma(1+\alpha)} \right). \quad (102)$$

(ii) If  $u = v = 0$  (maximal nonconservation of angular momentum) then

$$\Sigma = Q^{-1} \times \begin{pmatrix} e^{-i\pi\alpha}|w|^2(k^2/4) + e^{i\pi\alpha} & i \sin(\pi\alpha) \bar{w}k \\ i \sin(\pi\alpha) wk & e^{i\pi\alpha}|w|^2(k^2/4) + e^{-i\pi\alpha} \end{pmatrix}, \quad (103)$$



where

$$Q = |w|^2 \frac{k^2}{4} + 1 .$$

(iii) If  $\alpha = 1/2$  then

$$\Sigma = Q^{-1} \times \begin{pmatrix} -\imath - \imath(uv - |w|^2/4)k^2 + (u-v)k & -\imath\bar{w}k \\ -\imath wk & \imath + \imath(uv - |w|^2/4)k^2 + (u-v)k \end{pmatrix} , \quad (104)$$

where

$$Q = -1 + (uv - |w|^2/4)k^2 + \imath(u + v)k .$$

We conclude this section by giving (the kernel of) the full scattering operator in the angular representation,

$$S(k; \theta, \theta_0) = \frac{1}{2\pi} \sum_{m,n=-\infty}^{\infty} S(k; m, n) e^{\imath(m\theta - n\theta_0)} , \quad (105)$$

where

$$S(k; m, n) = \begin{cases} e^{\imath\pi(|m| - |m+\alpha|)} \delta_{mn} & \text{if } m, n \neq -1, 0 \\ \Sigma_{m+2, n+2} & \text{if } m, n = -1, 0 \end{cases}$$

(the shift by 2 is due to our labelling of rows and columns of  $\Sigma$ ). The double infinite sum can be performed in the sectors  $m, n \neq -1, 0$  (a known result borrowed from the pure A-B effect) and yields

$$\begin{aligned} S(k; \theta, \theta_0) &= \cos(\pi\alpha) \delta(\theta - \theta_0) + \frac{1}{2\pi} \sin(\pi\alpha) \frac{e^{-\imath(\theta - \theta_0)/2}}{\sin(\theta/2 - \theta_0/2)} \\ &+ \frac{1}{2\pi} \sum_{m,n=-1}^0 \left( \Sigma_{m+2, n+2} - e^{-(2m+1)\imath\pi\alpha} \delta_{mn} \right) e^{\imath(m\theta - n\theta_0)} . \end{aligned} \quad (106)$$

We recall that the differential cross-section in the plane is given by  $d\sigma(\theta)/d\theta = (2\pi/k) |S(k; \theta, \theta_0)|^2$ .

## 5 Conclusions

We have introduced and studied a five-parameter family of operators which describe a quantum mechanical particle interacting with a magnetic flux  $\alpha$  caused by an infinite thin material solenoid. We conjecture that in some well defined limit (when the thickness of the solenoid  $\rightarrow 0$  and its length  $\rightarrow \infty$ ) which should be largely independent on the details of approximating potentials (and which however goes beyond the scope of this paper) such a situation is described by a (singular) potential barrier and by a electromagnetic potential concentrated along the  $z$  axis (the magnetic field vanishes in the remaining region). One of the five parameters is just the value of the flux  $\alpha$  and the other four correspond to the strength of a singular potential barrier (sort of a combination of Dirac  $\delta$  and  $\delta'$ ) and can be interpreted as penetrability coefficients of the shielded solenoid.

A general operator of this family corresponds to an intricate mixture between the Aharonov-Bohm effect and the point interactions which is manifested more concretely via the mixing between the angular and the radial boundary conditions. It is interesting that the result we have obtained is richer than a simple superposition of the point interaction and the pure A-B effect. For instance, for a range of parameters there are two bound states possible while for the usual interactions with a support concentrated along the  $z$  axis and symmetric under the  $z$ -translations (or, equivalently, for a point interaction in two dimensions) there is always (except of the free case) exactly one bound state, and for the pure A-B effect there are no bound states at all.

In the present paper we have derived an explicit formula for the scattering matrix  $S(k; \theta, \theta_0)$  depending on the five parameters. Naturally, it would be of interest to examine the differential cross-section numerically, particularly its dependance on the parameters, and to deduce some physical consequences. This is what we plan to do separately.

It is a matter of experimental measurements (interference, scattering or in condensed matter) to establish which of the extensions in our family correspond to realizable models. On general grounds, one can distinguish some class of extensions by eg. symmetry requirements (conservation of the angular momentum), or postulating that there are no bound states (solenoid as a repulsive barrier).

After completing most of our work we became aware of [12] which has an overlap with some of our results. We have been also informed that a related preprint by R. Adami and A. Teta is in preparation.

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